

Research Article

A Generalization of Uniformly Extremely Convex Banach Spaces

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We discuss a new class of Banach spaces which are the generalization of uniformly extremely convex spaces introduced by Wulede and Ha. We prove that the new class of Banach spaces lies strictly between either the classes of k -uniformly rotund spaces and k -strongly convex spaces or classes of fully k -convex spaces and k -strongly convex spaces and has no inclusive relation with the class of locally k -uniformly convex spaces. We obtain in addition some characterizations and properties of this new class of Banach spaces. In particular, our results contain the main results of Wulede and Ha.

1. Introduction

Different uniformly convex spaces have been defined between the uniformly convex spaces [1] and the reflexivity of the Banach spaces [2–6]. In the previous paper [7] we introduce a new class of this type, namely, uniformly extremely convex spaces. This new class of Banach spaces lies strictly between either the classes of uniformly convex spaces and strongly convex spaces or the classes of fully k -convex spaces and strongly convex spaces.

Here we consider another new class of this type, namely, k -uniformly extremely convex spaces, as a generalization of uniformly extremely convex spaces and discuss its relation to the drop property, the k -uniformly rotund spaces, the full k -convex spaces, the k -strongly convex spaces, the nearly uniformly convex spaces, and k -nearly uniformly convex spaces. We also give some characterizations of k -uniformly extremely convex spaces and find that this new class of Banach spaces has the following features:

- (1) 1-uniformly extremely convex spaces (*indeed* lower case) coincide with uniformly extremely convex spaces;
- (2) k -uniformly extremely convex spaces possess the drop linebreak property;

- (3) k -uniformly extremely convex spaces are $(k + 1)$ -uniformly extremely convex spaces, but the converse implication is not true.

Throughout this paper X denotes an infinite-dimensional real Banach space with the norm $\|\cdot\|$. The symbol X^* denotes the dual of the space X . $U(X)$ and $S(X)$ denote the closed unit ball and the unit sphere of X , respectively. The symbol $S(X^*)$ denotes the unit sphere of X^* . The symbol $\sigma(X, X^*)$ denotes the weak topology of X .

Let x_1, x_2, \dots, x_{k+1} be norm-1 elements in Banach spaces X . The k -dimensional volume enclosed by x_1, x_2, \dots, x_{k+1} is given by

$$V(x_1, x_2, \dots, x_{k+1}) = \sup \left\{ \left| \begin{vmatrix} 1 & 1 & \cdots & 1 \\ f_1(x_1) & f_1(x_2) & \cdots & f_1(x_{k+1}) \\ \vdots & \vdots & \ddots & \vdots \\ f_k(x_1) & f_k(x_2) & \cdots & f_k(x_{k+1}) \end{vmatrix} \right| : f_1, \dots, f_k \in S(X^*) \right\}. \quad (1)$$

Here, and throughout the sequel, the symbol $|\cdot|$ denotes the determinant.

Sullivan [6] has introduced the k -uniformly rotund (kUR) spaces and locally k -uniformly rotund ($LkUR$) spaces. Fan and Glicksberg [2] have introduced the fully k -convex (kR) Banach spaces. It is well known that kUR and kR spaces imply reflexivity. About kUR and kR spaces, we have the following chain of implications [2, 6, 8]:

$$\begin{aligned} UR &= 1UR \implies \dots \implies kUR \implies (k+1)UR; \\ 2R &\implies \dots \implies kR \implies (k+1)R; \\ LUR &= L1UR \implies \dots \implies LkUR \implies L(k+1)UR. \end{aligned} \quad (2)$$

A Banach space X is said to be a kUR space ($k \geq 1$) [6] if, for any $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$ such that, for all norm-1 elements x_1, x_2, \dots, x_{k+1} and $\|x_1 + x_2 + \dots + x_{k+1}\| > (k+1) - \delta$, we have $V(x_1, x_2, \dots, x_{k+1}) < \epsilon$.

A Banach space X is said to be a kR space ($k \geq 2$) [2] if, for any sequence $\{x_n\}$ in X such that $\lim_{n_1, \dots, n_k \rightarrow \infty} (1/k) \|x_{n_1} + x_{n_2} + \dots + x_{n_k}\| = 1$, then $\{x_n\}$ is a Cauchy sequence in X .

A point $x_0 \in S(X)$ is said to be a denting point of $U(X)$ [8] if $x_0 \notin \overline{\text{co}}(M(x_0, \epsilon))$ for all $\epsilon > 0$, where $M(x_0, \epsilon) = \{y : y \in U(X), \|y - x_0\| \geq \epsilon\}$.

Huff [3] has introduced the nearly uniformly convex (NUC) spaces as a generalization of uniformly convex Banach spaces and showed that the NUC spaces are equivalent to reflexive spaces possessing the uniform Kadec-Klee property. The local version of NUC spaces, namely, locally nearly uniformly convex (LNUC), was studied by Kutzarova and Lin [9]. Kutzarova [4] introduced the k -nearly uniformly convex ($kNUC$) spaces as a generalization of nearly uniformly convex Banach spaces. In [4, 9], it is pointed out that $NUC \implies LNUC$ and $kNUC \implies NUC$ for every $k \geq 2$.

A Banach space X is said to be a NUC [3] space if, for any $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$ such that, for any sequence $\{x_n\} \subset U(X)$, $\text{sep}(x_n) > \epsilon$, we have $\text{co}(\{x_n\}) \cap (1 - \delta)U(X) \neq \emptyset$, where $\text{sep}(x_n) = \inf\{\|x_n - x_m\| : n \neq m\}$ and $\text{co}(\{x_n\})$ means the convex hull of $\{x_n\}$.

A Banach space X is said to be a LNUC [9] space if, for any norm-1 element x and $\epsilon > 0$, there exists a $\delta = \delta(\epsilon, x) > 0$ such that, for any sequence $\{x_n\} \subset U(X)$, $\text{sep}(x_n) > \epsilon$, we have $\text{co}(\{x\} \cup \{x_n\}) \cap (1 - \delta)U(X) \neq \emptyset$, where $\text{co}(\{x\} \cup \{x_n\})$ means the convex hull of $\{x\}$ and $\{x_n\}$.

A Banach space X is said to be a $kNUC$ [4] space, if for any $\epsilon > 0$ there exists a $0 < \delta(\epsilon) < 1$ such that, for any sequence $\{x_n\} \subset U(X)$, $\text{sep}(x_n) > \epsilon$, there are indices $\{n_i\}$ and scalars $\lambda_i \geq 0$, $i = 1, 2, \dots, k$, with $\sum_{i=1}^k \lambda_i = 1$ so that $\|\sum_{i=1}^k \lambda_i x_{n_i}\| \leq 1 - \delta$.

Singer [10] has introduced the k -strictly convex spaces. It is well known that k -strictly convex spaces are $(k+1)$ -strictly convex spaces; 1-strictly convex spaces (*indeed* lower case) coincide with strictly convex spaces; kR spaces are k -strictly convex spaces and have the drop property. Wu and Li [11] have introduced the strongly convex spaces. Wulede and Wu [12] introduced the k -strongly convex spaces as a generalization of strongly convex Banach spaces and gave an equivalent definition of k -strongly convex spaces (see Theorem 5 in [13]).

It is well known that k -strongly convex spaces are k -strictly convex spaces; 1-strongly convex spaces (*indeed* lower case) coincide with strongly convex spaces; k -strongly convex spaces are $(k+1)$ -strongly convex spaces, but the converse implication is not true.

A Banach space X is said to be a k -strictly convex space [10] if, for all norm-1 elements x_1, x_2, \dots, x_{k+1} such that $\|\sum_{i=1}^{k+1} x_i\| = k+1$, then x_1, x_2, \dots, x_{k+1} are linearly dependent.

A Banach space X is said to be a strongly convex space [11] if, for any $x \in S(X)$, $\{x_n\} \subset S(X)$ and for a certain functional $f \in S_x$ such that $f(x_n) \rightarrow 1$ ($n \rightarrow \infty$), then $\|x_n - x\| \rightarrow 0$ ($n \rightarrow \infty$), where $S_x = \{f : f \in S(X^*), f(x) = 1\}$.

A Banach space X is said to be a k -strongly convex space [12] if, for any norm-1 element x , $\epsilon > 0$ and for any functional $f \in S_x$, there is a $\delta(x, f, \epsilon) > 0$ such that, for all norm-1 elements x_1, \dots, x_k and $f(x + x_1 + \dots + x_k) > (k+1) - \delta$, we have $V(x, x_1, \dots, x_k) < \epsilon$.

Rolewicz [14] has defined the norm $\|\cdot\|$ to have the drop property, if for every closed set $C \subset X$ disjoint from $U(X)$ there exists $x \in C$ such that $D(x, U(X)) \cap C = \{x\}$, where the set $D(x, U(X))$, the convex hull of x and $U(X)$, is called the drop generated by $x \notin U(X)$.

Lemma 1 (Kadec-Klee property). *If any $x \in S(X)$, $\{x_n\} \subset S(X)$ such that $x_n \xrightarrow{w} x$, $n \rightarrow \infty$, and $\|x_n\| \rightarrow \|x\|$, $n \rightarrow \infty$, then $\|x_n - x\| \rightarrow 0$, $n \rightarrow \infty$, where $x_n \xrightarrow{w} x$, $n \rightarrow \infty$, means that $f(x_n) \rightarrow f(x)$, $n \rightarrow \infty$, for all $f \in X^*$.*

Lemma 2 (Montesinos [15]). *Let X be a Banach space. Then X has the drop property if and only if X is reflexive and has the Kadec-Klee property.*

Lemma 3 (Nan and Wang [16]). *X is k -strictly convex space if and only if, for any $f \in S(X^*)$, one has $\dim A_f \leq k$, where $A_f = \{x : x \in S(X), f(x) = 1\}$.*

Lemma 4 (Wulede and Wu [12], Zhang and Fang [17]). *Let X be a Banach space.*

- (i) *If X is k -strongly convex, then X is k -strictly convex and has the Kadec-Klee property.*
- (ii) *If X is reflexive, k -strictly convex and has the Kadec-Klee property, then X is k -strongly convex.*
- (iii) *If X is k -strongly convex, $\{x_n\} \subset U(X)$, $f \in S(X^*)$, and $f(x_n) \rightarrow 1$, $n \rightarrow \infty$, then $\text{dist}(x_n, A_f) \rightarrow 0$, $n \rightarrow \infty$.*

Lemma 5 (Zhang and Fang [17]). *X is k -strongly convex if and only if, for any $x \in S(X)$ and $f \in S_x$, $\epsilon > 0$, there exist $\delta > 0$ and a compact set $C \subset X$ with $\dim C \leq k$ such that $F(f, \delta) \subset \{x : x \in X, d(x, C) < \epsilon\}$, where the set $F(f, \delta) = \{x : x \in U(X), f(x) \geq 1 - \delta\}$ is the slice generated by f and δ .*

2. k -Uniformly Extremely Convex Spaces and Drop Property

Definition 6 (see [7]). A Banach space X is said to be a uniformly extremely convex space if, for any sequences

$\{x_n\}, \{y_n\}$ consisting of elements of norm-1 and for a certain functional f of norm-1, $\lim_{n \rightarrow \infty} f(x_n) = \lim_{n \rightarrow \infty} f(y_n) = 1$ holds; then $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$.

On the base of uniformly extremely convex spaces, now we introduce the notion of k -uniformly extremely convex spaces as a generalization of uniformly extremely convex spaces.

Definition 7. A Banach space X is said to be a k -uniformly extremely convex space if, for any sequences $\{x_n^{(1)}\}, \dots, \{x_n^{(k+1)}\}$ consisting of elements of norm-1 and for a certain functional f of norm-1, $\lim_{n \rightarrow \infty} f(x_n^{(1)}) = \dots = \lim_{n \rightarrow \infty} f(x_n^{(k+1)}) = 1$ holds; then $\lim_{n \rightarrow \infty} V(x_n^{(1)}, \dots, x_n^{(k+1)}) = 0$.

We give first a simple result which shows that the notion of k -uniformly extremely convex space is “coherent.”

Theorem 8. If X is k -uniformly extremely convex space, then X is $(k+1)$ -uniformly extremely convex space.

Proof. If, for any sequences $\{x_n^{(1)}\}, \dots, \{x_n^{(k+2)}\}$ consisting of elements of norm-1 and for a certain functional f of norm-1, $\lim_{n \rightarrow \infty} f(x_n^{(1)}) = \dots = \lim_{n \rightarrow \infty} f(x_n^{(k+2)}) = 1$ holds, then, for all $1 \leq j \leq k+2$, we have $\lim_{n \rightarrow \infty} V(x_n^{(1)}, \dots, x_n^{(j-1)}, x_n^{(j+1)}, \dots, x_n^{(k+2)}) = 0$ by the assumption that X is k -uniformly extremely convex space. Furthermore, by the properties of determinant we have

$$\begin{aligned} & \lim_{n \rightarrow \infty} V(x_n^{(1)}, \dots, x_n^{(k+2)}) \\ & \leq \lim_{n \rightarrow \infty} \sum_{j=2}^{k+1} V(x_n^{(1)}, \dots, x_n^{(j-1)}, x_n^{(j+1)}, \dots, x_n^{(k+2)}) = 0; \end{aligned} \quad (3)$$

this shows that X is $(k+1)$ -uniformly extremely convex space. \square

Now we give a simple but useful lemma. By using this lemma we can prove that any k -uniformly extremely convex space has the drop property. And the fact that k -uniformly extremely convex spaces include k -strongly convex spaces can be easily found. We also show that 1-uniformly extremely convex spaces coincide with uniformly extremely convex spaces by using this lemma.

Lemma 9. X is k -uniformly extremely convex if and only if, for any $\epsilon > 0$, $f \in S(X^*)$, there exists a $\delta(\epsilon) > 0$ such that, for all norm-1 elements x_1, \dots, x_{k+1} and $f(\sum_{i=1}^{k+1} x_i) > (k+1) - \delta$, one has $V(x_1, \dots, x_{k+1}) < \epsilon$.

Proof.

Necessity. Suppose the contrary. Then there exist $\epsilon_0 > 0$, $f_0 \in S(X^*)$ and $\{x_i\}_{i=1}^{k+1} \subset S(X)$ such that, for any $\delta = 1/n$, $n \in \mathbb{N}$, we have $f_0(\sum_{i=1}^{k+1} x_i) > (k+1) - 1/n$, but $V(x_1, \dots, x_{k+1}) \geq \epsilon_0$. Take $x_n^{(i)} = x_i$ ($i = 1, \dots, k+1$); then $\{x_n^{(i)}\}_{i=1}^{k+1} \subset S(X)$ and $k+1 - 1/n < f_0(\sum_{i=1}^{k+1} x_n^{(i)}) \leq k+1$. It follows that

$\lim_{n \rightarrow \infty} f_0(x_n^{(i)}) = 1$. On the other hand, by the definition of the k -uniformly extremely convex space, we have $V(x_1, \dots, x_{k+1}) \rightarrow 0$; this contradicts the statement that $V(x_1, \dots, x_{k+1}) \geq \epsilon_0$.

Sufficiency. If, for any sequences $\{x_n^{(1)}\}, \dots, \{x_n^{(k+1)}\}$ consisting of elements of norm-1 and for a certain functional f of norm-1, $\lim_{n \rightarrow \infty} f(x_n^{(1)}) = \dots = \lim_{n \rightarrow \infty} f(x_n^{(k+1)}) = 1$ holds, then $\lim_{n \rightarrow \infty} f(\sum_{i=1}^{k+1} x_n^{(i)}) = k+1$. Therefore, for any $\delta > 0$, there exists an integer $N_0 \in \mathbb{N}$ such that, for all $n \geq N_0$, inequality $f(\sum_{i=1}^{k+1} x_n^{(i)}) > (k+1) - \delta$ holds. For any $\epsilon > 0$, by the conditions given in Lemma 9, we have $V(x_n^{(1)}, \dots, x_n^{(k+1)}) < \epsilon$; this means that $\lim_{n \rightarrow \infty} V(x_n^{(1)}, \dots, x_n^{(k+1)}) = 0$. \square

Remark 10. 1-uniformly extremely convex space (indeed lower case) coincides with uniformly extremely convex space.

In fact, by Lemma 9 we know that X is 1-uniformly extremely convex space if and only if, for any $\epsilon > 0$, $f \in S(X^*)$, there exists a $\delta(\epsilon, f) > 0$ such that, for any norm-1 elements x, y and $f(x+y) > 2 - \delta$, we have

$$\begin{aligned} V(x, y) &= \sup \left\{ \left| \begin{vmatrix} 1 & 1 \\ f(x) & f(y) \end{vmatrix} \right| : f \in S(X^*) \right\} \\ &= \sup_{f \in S(X^*)} |f(x) - f(y)| = \|x - y\| < \epsilon, \end{aligned} \quad (4)$$

and also if and only if X is uniformly extremely convex space.

Theorem 11. X is k -uniformly extremely convex space if and only if X is k -strictly convex space and has the drop property.

Proof.

Necessity. Suppose that X is k -uniformly extremely convex space; by the definition of k -strongly convex space and a condition which characterizes k -uniformly extremely convex space in Lemma 9, it is easy to see that X is k -strongly convex space. From Lemma 4(i), we know that X is k -strictly convex space and has the Kadec-Klee property.

Now we are going to prove that X has the drop property. In fact, from Lemma 2, it is sufficient to prove that X is reflexive. Suppose that X is not reflexive. Using the well-known James' theorem, for each $0 < \epsilon < 1$, we can choose $0 < \theta < 1$ so that $\theta > 1 - \delta(\epsilon)/(k+1)$ and $\theta^k > \epsilon$, and $\{x_1\}, \dots, \{x_{k+1}\} \subset S(X)$, $\{x_1^*\}, \dots, \{x_{k+1}^*\} \subset S(X^*)$ so that

$$x_j^*(x_i) = \begin{cases} \theta & \text{if } j \leq i \\ 0 & \text{if } j > i. \end{cases} \quad (5)$$

Here $\delta(\epsilon)$ is the function required in Lemma 9.

Now we have that

$$x_1^*(x_1 + x_2 + \dots + x_{k+1}) = (k+1)\theta > (k+1) - \delta(\epsilon). \quad (6)$$

On the other hand it is easy to check that

$$V(x_1, \dots, x_{k+1}) \geq \begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_2^*(x_1) & x_2^*(x_2) & \cdots & x_2^*(x_{k+1}) \\ \vdots & \vdots & \ddots & \vdots \\ x_{k+1}^*(x_1) & x_{k+1}^*(x_2) & \cdots & x_{k+1}^*(x_{k+1}) \end{vmatrix} = \theta^k > \epsilon, \quad (7)$$

which gives the required contradiction.

Sufficiency. From the assumption that X is k -strictly convex space and has the drop property, we can deduce, by Lemma 4(ii), that X is k -strongly convex space and reflexive. Observing the definition of k -strongly convex space and a condition which characterizes k -uniformly extremely convex space in Lemma 9, by the reflexivity of X , it is easy to see that X is k -uniformly extremely convex space. \square

Corollary 12 (see [7]). *X is uniformly extremely convex space if and only if X is strictly convex space and has the drop property.*

Noticing the procedure of proving Theorem 11 we can deduce the following.

Corollary 13. *If X is k -uniformly extremely convex space, then X is k -strongly convex space.*

Now we are going to show that the converse to Corollary 13 is not true. In [12], it is proved that $LkUR$ spaces are k -strongly convex spaces. In general, $LkUR$ spaces need not be reflexive since $L1UR$ is just the usual definition of LUR space [18, 19]. It follows that there exists a k -strongly convex space X which is not reflexive. Hence X is not a k -uniformly extremely convex space since X is not reflexive.

Corollary 14. *X is k -uniformly extremely convex space if and only if X is reflexive and, for any $x \in S(X)$ and $f \in S_x$, $\epsilon > 0$, there exist $\delta > 0$ and a compact set $C \subset X$ with $\dim C \leq k$ such that $F(f, \delta) \subset \{x : x \in X, d(x, C) < \epsilon\}$, where the set $F(f, \delta) = \{x : x \in U(X), f(x) \geq 1 - \delta\}$ is the slice generated by f and δ .*

Proof. It is immediate from Corollary 13, Theorem 11, and Lemmas 2, 4, and 5. \square

Theorem 15. *X is k -uniformly extremely convex space if and only if X is reflexive and, for any $f \in S(X^*)$, one has $\dim A_f \leq k$, $A_f \cap \overline{\text{co}}(U(X) \setminus V_{A_f}) = \emptyset$, where the set V_{A_f} , which includes set A_f , is arbitrary open set with regard to norm topology $(X, \|\cdot\|)$.*

Proof.

Necessity. Suppose that X is k -uniformly extremely convex space; then by Theorem 11 we know that X is k -strictly convex space and reflexive. For any $f \in S(X^*)$, by the reflexivity of X , there exists $x \in S(X)$ such that $f(x) = 1$; hence $x \in A_f$.

Combining the fact that X is k -strictly convex space with Lemma 3 we can deduce that $\dim A_f \leq k$.

Now we are going to prove the equality $A_f \cap \overline{\text{co}}(U(X) \setminus V_{A_f}) = \emptyset$.

Firstly, we will prove that, for any $z \notin V_{A_f}$ and every open set V_{A_f} (where $V_{A_f} \supset A_f$) with regard to norm topology $(X, \|\cdot\|)$, there exists a scalar $r > 0$ such that $\text{dist}(z, A_f) \geq r$.

Noticing that A_f is compact set in X , for any $z \notin V_{A_f}$, we can deduce that there exists $x \in A_f$ such that $\|x - z\| = \text{dist}(z, A_f) = r_z$. Now we claim that there exists minimum value of r_z denoted by r , such that $\text{dist}(z, A_f) \geq r$ for any $z \notin V_{A_f}$. In fact, if r_z does not have minimum value, then $1/n$ is impossible to be minimum value for any integer n . Hence, there exist $z_n \notin V_{A_f}$ and $x_n \in A_f$ such that $\|x_n - z_n\| = \text{dist}(z_n, A_f) < 1/n$. Since A_f is compact, the above sequence $\{x_n\}$ has the convergent subsequence; without loss of generality and letting the convergent subsequence be $\{x_n\}$ itself, then $x_n \rightarrow x_0$, $x_0 \in A_f$. Noticing that $\|x_n - z_n\| = \text{dist}(z_n, A_f) < 1/n$, we can deduce that $z_n \rightarrow x_0$, $x_0 \in A_f \subset V_{A_f}$.

On the other hand, combining the fact that $X \setminus V_{A_f}$ is closed set with $z_n \in X \setminus V_{A_f}$, $z_n \rightarrow x_0$, we can deduce that $x_0 \in X \setminus V_{A_f}$. This contradicts $x_0 \in V_{A_f}$.

Secondly, we will prove that for V_{A_f} there exists a scalar $m > 0$ such that the inequality $f(x) > f(y) + m$ holds for all $x \in A_f$ and $y \in U(X) \setminus V_{A_f}$.

If the above inequality is not true, then there exists $y_n \in U(X) \setminus V_{A_f}$ such that $f(y_n) \rightarrow f(x) = 1$, $n \rightarrow \infty$. By the condition given in Theorem 15, Corollary 13, and Lemma 4(iii), we have $\text{dist}(y_n, A_f) \rightarrow 0$, $n \rightarrow \infty$. On the other hand, by $y_n \in U(X) \setminus V_{A_f}$, we can deduce that $\text{dist}(y_n, A_f) \not\rightarrow 0$, $n \rightarrow \infty$; this contradicts the statement that $\text{dist}(y_n, A_f) \rightarrow 0$, $n \rightarrow \infty$. Hence we have

$$f(x) - m \geq \sup \{f(y) : y \in U(X) \setminus V_{A_f}\} = \sup \{f(y) : y \in \overline{\text{co}}(U(X) \setminus V_{A_f})\}; \quad (8)$$

this shows that $x \notin \overline{\text{co}}(U(X) \setminus V_{A_f})$. By the arbitrary of $x \in A_f$, we can deduce that $A_f \cap \overline{\text{co}}(U(X) \setminus V_{A_f}) = \emptyset$.

Sufficiency. By Lemmas 2 and 3, Theorem 11, and the condition given in Theorem 15, only we need to prove that X has the Kadec-Klee property. Let $x \in S(X)$, $\{x_n\}_{n=1}^\infty \subset S(X)$, and $x_n \xrightarrow{w} x$, $n \rightarrow \infty$. By the well-known James' theorem, there exists $f \in S(X^*)$ such that $f(x) = 1$; it follows that $x \in A_f$.

Case 1. If $\{x_n\}_{n=1}^\infty \cap A_f = \emptyset$, then $\{x_n\}_{n=1}^\infty$ is relatively compact. Otherwise, every point of A_f is not accumulation point of $\{x_n\}_{n=1}^\infty$. Hence, for any $x \in A_f$ there exists $\epsilon_0 > 0$ such that $\{y \in X : \|y - x\| < \epsilon_0\}$ does not contain any point of $\{x_n\}_{n=1}^\infty$. We construct an open set $V_{A_f} = \bigcup_{x \in A_f} \{y \in X : \|y - x\| < \epsilon_0\}$ with regard to norm topology $(X, \|\cdot\|)$; then $A_f \subset V_{A_f}$ and $V_{A_f} \cap \{x_n\}_{n=1}^\infty = \emptyset$. Since $\overline{\text{co}}(U(X) \setminus V_{A_f})$ is bounded closed convex set with regard to norm topology

$(X, \|\cdot\|)$, $\overline{\text{co}}^w(U(X) \setminus V_{A_f}) = \overline{\text{co}}(U(X) \setminus V_{A_f})$. Noticing that $\overline{\text{co}}^w(U(X) \setminus V_{A_f})$ is bounded set with regard to weak topology $\sigma(X, X^*)$, we know that $\overline{\text{co}}^w(U(X) \setminus V_{A_f})$ is compact set with regard to weak topology $\sigma(X, X^*)$. Hence there is a function $g \in X^*$ which separates A_f and $\overline{\text{co}}(U(X) \setminus V_{A_f})$; that is, there is a scalar $l > 0$ such that $g(A_f) - l > \sup g(\overline{\text{co}}(U(X) \setminus V_{A_f}))$. Evidently, $\{x_n\}_{n=1}^\infty \subset \overline{\text{co}}(U(X) \setminus V_{A_f})$; it follows that $g(A_f) - g(\{x_n\}_{n=1}^\infty) > l$. This contradicts the assumption that $x_n \xrightarrow{w} x$, $n \rightarrow \infty$.

Case 2. If $\{x_n\}_{n=1}^\infty \cap A_f \neq \emptyset$, then $(\{x_n\}_{n=1}^\infty \setminus A_f) \cap A_f = \emptyset$. According to Case 1 we know that $\{x_n\}_{n=1}^\infty \setminus A_f$ is relatively compact set. Hence $\{x_n\}_{n=1}^\infty \cap A_f$ is compact set since A_f is bounded closed convex set in certain finite dimensional subspace of X . On the other hand, it is obvious that $\{x_n\}_{n=1}^\infty = (\{x_n\}_{n=1}^\infty \setminus A_f) \cup (\{x_n\}_{n=1}^\infty \cap A_f)$; hence $\overline{\{x_n\}_{n=1}^\infty} = (\overline{\{x_n\}_{n=1}^\infty \setminus A_f}) \cup (\overline{\{x_n\}_{n=1}^\infty \cap A_f})$. This shows that $\{x_n\}_{n=1}^\infty$ is relatively compact.

Consequently, in Cases 1 and 2, we always conclude that $\{x_n\}_{n=1}^\infty$ is relatively compact. Furthermore, by the assumption that $x_n \xrightarrow{w} x$, $n \rightarrow \infty$, we can deduce that $\|x_n - x\| \rightarrow 0$, $n \rightarrow \infty$. This completes the proof that X has the Kadec-Klee property. \square

In particular, considering the special case of Theorem 15 when $k = 1$, we obtained Theorem 2.5 in [7] as a corollary.

Corollary 16. *X is uniformly extremely convex space if and only if X is reflexive and for any $f \in S(X^*)$, one has $\dim A_f = 1$, $A_f \cap \overline{\text{co}}(U(X) \setminus V_{A_f}) = \emptyset$, where the set V_{A_f} , which includes set A_f , is arbitrary open set with regard to norm topology $(X, \|\cdot\|)$. In other words, X is uniformly extremely convex space if and only if X is reflexive and every point of $S(X)$ is a denting point of $U(X)$.*

To show that the converse to Theorem 8 is not true, we consider the following example.

Example 17. There exists a k -uniformly extremely convex space X which is not a $(k-1)$ -uniformly extremely convex space.

Let $k \geq 2$ be an integer, and let $i_1 < i_2 < \dots < i_k$. For each $x = (a_1, a_2, \dots) \in l_2$, define

$$\|x\|_{i_1, \dots, i_k}^2 = \left(\sum_{j=1}^k |a_{i_j}| \right)^2 + \sum_{i \neq i_1, \dots, i_k} a_i^2. \quad (9)$$

From [20] we know that $X_{i_1, \dots, i_k} = (l_2, \|\cdot\|_{i_1, \dots, i_k})$ is a k UR space. It is easy to see that k UR space is k -uniformly extremely convex space from the definition of k UR space and a condition which characterizes k -uniformly extremely convex space in Lemma 9. Hence X_{i_1, \dots, i_k} is a k -uniformly extremely convex space. It follows from Theorem 11 that X_{i_1, \dots, i_k} is a k -strictly convex space but is not a $(k-1)$ -strictly convex space that follows from [16]. Hence X_{i_1, \dots, i_k} is not a $(k-1)$ -uniformly extremely convex space.

3. The Relations between k -Uniformly Extremely Convex Space and Various Other Types of Convex Space

Now we give a list of examples to distinguish k -uniformly extremely convex spaces from k R, k UR, k NUC, and NUC spaces.

- (i) We are ready now to distinguish k -uniformly extremely convex and k R spaces.

Since k R spaces are k -strictly convex spaces and have the drop property, it follows from Theorem 11 that k R spaces are k -uniformly extremely convex, but the converse is not true.

Example 18. There exists a k -uniformly extremely convex space X which is not a k R space for every $k \geq 2$.

Let $k \geq 2$ be an integer, and let $i_1 < i_2 < \dots < i_k$. For each $x = (a_1, a_2, \dots) \in l_2$, define

$$\|x\|_{i_1, \dots, i_k}^2 = \left(\sum_{j=1}^k |a_{i_j}| \right)^2 + \sum_{i \neq i_1, \dots, i_k} a_i^2, \quad (10)$$

and let $X_{i_1, \dots, i_k} = (l_2, \|\cdot\|_{i_1, \dots, i_k})$. For $x \in l_2$, let $\|x\|_k = \sup_{i_1 < i_2 < \dots < i_k} \|x\|_{i_1, \dots, i_k}$, $X_k = (l_2, \|\cdot\|_k)$. It follows from [20] that X_k is a k UR space but is not a k R space. Hence X_k is a k -uniformly extremely convex space since X is a k UR space.

- (ii) We are ready now to distinguish k -uniformly extremely convex and k UR spaces.

Example 19. For all $k \geq 1$, there exists a k -uniformly extremely convex space X which is not a k UR space.

Let $E = (l_2, \|\cdot\|)$; for $x = (a_1, a_2, \dots) \in E$, define

$$\|x\|^2 = \left\{ |a_1| + (a_2^2 + a_3^2 + \dots)^{1/2} \right\}^2 + \left\{ \left(\frac{a_2}{2} \right)^2 + \dots + \left(\frac{a_n}{n} \right)^2 + \dots \right\}^2. \quad (11)$$

It follows from [2] that $X = (\sum \oplus E)_{l_2}$ is a 2R space; furthermore, $X = (\sum \oplus E)_{l_2}$ is a k -uniformly extremely convex space but is not a k UR space [20].

- (iii) We are ready now to distinguish k -uniformly extremely convex and L kUR spaces.

We consider a nonreflexive L kUR space X . Then X is not a k -uniformly extremely convex space since X is not reflexive. On the other hand, we consider Example 19; then $X = (\sum \oplus E)_{l_2}$ is a 2R space and it follows that X is a k -uniformly extremely convex space for all $k \geq 1$. But X is not a L kUR space that follows from [21].

- (iv) We are ready now to distinguish k -uniformly extremely convex spaces and NUC or k NUC spaces.

Example 20. For all $k \geq 1$, there exists a k -uniformly extremely convex space X which is neither a NUC nor a k NUC space for all $k \geq 2$.

Let $(X, \|\cdot\|)$ be the l_2 -sum of $\{l_n, n \geq 2\}$; then $(X, \|\cdot\|)$ is a 2R space with normalized basis $\{e_n\}$. Define, $\forall x = \sum_{n=1}^{\infty} a_n e_n \in X$,

$$\|x\| = \left\{ \left(|a_1| + \left\| \sum_{n=2}^{\infty} a_n e_n \right\| \right)^2 + \sum_{n=2}^{\infty} \left(\frac{a_n}{n} \right)^2 \right\}^{1/2}. \quad (12)$$

By Theorem 4 in [9], we know that $(X, \|\cdot\|)$ is a 2R space but is not a LNUC space. It follows that X is a k -uniformly extremely convex space for all $k \geq 1$ but is neither a NUC nor a k NUC space for all $k \geq 2$.

Remark 21. (i) The class of k -uniformly extremely convex spaces lies strictly between the classes of k UR spaces and the k -strongly convex spaces.

(ii) The class of k -uniformly extremely convex spaces lies strictly between the classes of k R spaces and the class of k -strongly convex spaces.

(iii) The class of k -uniformly extremely convex spaces has no inclusive relation with the class of Lk UR spaces.

In particular, considering the special case of Remark 21 when $k = 1$, we obtained the main conclusions of [7], that is, Remarks 3.5 and 3.7 in [7].

Competing Interests

The authors declare that they have no competing interests.

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